ORIENTATIONS OF FORMAL GROUPS

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Abstract. In his 2018 monograph on elliptic cohomology, Lurie developed the theory of orientations of formal groups, a powerful new tool in chromatic homotopy theory that strengthens the classical parameterization of complex-oriented cohomology theories by formal groups and extends it to the $\mathbb{E}_\infty$ setting. The goal of this talk is to provide a conceptually accessible introduction to orientations: what they are, why they're useful, and how they can be interpreted algebraically and geometrically.

1. A recap of the classical story

We have a standard map

$$\begin{array}{ccc} \mathbb{CP}^1 & \xhookrightarrow{\ i\ } & \mathbb{CP}^\infty \\ \cong & & \wr \\ S^2 & \xrightarrow{\ i\ } & K(\mathbb{Z},2) \end{array}$$

where $[i] = 1 \in \pi_2 K(\mathbb{Z},2)$, i.e. it is the canonical generator. Now let R be a ring spectrum, so we have a unit element $1 \in \pi_0(R)$, and take $R^2(i)$:

$$\begin{array}{ccc} \mathrm{Map}_*(S^2, \Omega^{\infty-2}R) & \xleftarrow{\ i^*\ } & \mathrm{Map}_*(K\mathbb{Z},2), \Omega^{infty-2}R) \\ \wr & & \wr \\ \mathrm{Map}_*(S^0, \Omega^\infty R) & \longleftarrow & \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \Omega^\infty R) \end{array}$$

The left equivalence comes from suspension-loop adjunction, while the right equivalence comes from delooping. We have $\mathrm{Map}_*(S^0, \Omega^\infty R) \simeq \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}, \Omega^\infty R)$ because $\mathbb{Z}$ is the free $\mathbb{E}_1$-space on S^0, so we get a map

$$(*) \qquad \mathrm{Map}_{\mathbb{E}_1}(\mathbb{Z}, \Omega^\infty R) \leftarrow \mathrm{Map}_{\mathbb{E}_2}(\mathbb{Z}, \Omega^\infty R)$$

which coincides with the evident forgetful map, as I showed in my talk last semester (c.f. [1]).

Definition 1.1. A *complex orientation* is an element $c_1 \in R^2(\mathbb{CP}^\infty)$ restricting to 1 along the natural map $R^2(\mathbb{CP}^\infty) \to R^2(S^2) \cong \pi_0 R$.

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Since this restriction map is equivalent to $(*)$, this is a kind of "strictification" of the unit of R. Unlike S^2, $\mathbb{CP}^\infty$ is an abelian group; in fact, it is the free "strict abelian group" on the pointed space S^2. As a consequence, its cohomology is a formal group. The graded version of this formal group is $\mathrm{Spf}\, R^*(\mathbb{CP}^\infty)$ with adic topology coming from the isomorphism $R^*(\mathbb{CP}^\infty) \cong (\pi_*(R))[[c_1]]$. By applying the forgetful functor $\mathrm{FGroup}(\pi_*(R)) \to \mathrm{FGroup}(\pi_0(R))$, we get a formal group $\widehat{\mathbb{G}}_R^{Q_0}$ over $\pi_0 R$.

Definition 1.2. The formal group $\widehat{\mathbb{G}}_R^{Q_0}$ is called the *classical Quillen formal group.*

Remark 1.3. If there is an invertible element $u \in \pi_2 R$, we can write $\widehat{\mathbb{G}}_R^{Q_0} = \mathrm{Spf}\, R^0(\mathbb{CP}^\infty)$, where the adic topology is generated by $u^{-1}c_1$.

One can also construct $\widehat{\mathbb{G}}_R^{Q_0}$ more "intrinsically" if R is weakly 2-periodic. In this case, $R_0(\mathbb{CP}^\infty)$ is a smooth one-dimensional cocommutative Hopf algebra, and its cospectrum is $\widehat{\mathbb{G}}_R^{Q_0}$.

2. Quillen According to Lurie

I'll set up some technical definitions to lift the above into the realm of spectral algebraic geometry. The material in this section, as well as the remainder of the talk, originates from [2].

Definition 2.1. An *adic* $\mathbb{E}_\infty$ *R-algebra* is an $\mathbb{E}_\infty$ R-algebra A with an adic topology on $\pi_0 A$. Such a ring is called *smooth* if, roughly speaking, its homotopy ring is a completed symmetric algebra on a finite-rank projective module. (To be precise, it should be the complete symmetric algebra on a $\pi_0 R$-module, tensored up to $\pi_* R$.) Equivalently, A is smooth if its dual is a smooth coalgebra, meaning $\pi_0(A^\vee) \cong \Gamma^*(M)$.

By the functor-of-points construction, we have a fully faithful embedding

$$\begin{array}{ccc} \mathrm{cCAlg}_R^{sm} & \xrightarrow{\mathrm{cSpec}} & \mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathrm{Spaces}) \\ \wr\| & \nearrow_{\mathrm{Spf}} & \\ \mathrm{CAlg}_R^{ad,sm} & & \end{array}.$$

An object in the image of this embedding is called a *formal hyperplane.* Basically, these are smooth affine objects in the category of formal spectral Deligne-Mumford stacks.

Definition 2.2. A *formal group* over R is an abelian group object in the category of formal hyperplanes, i.e. a product-preserving functor $\mathrm{Lat}^{op} \to \mathrm{Hyp}(R)$.

(Here, Lat is the category of finitely-generated free abelian groups.) What I've called an abelian group object here is sometimes called a "strict abelian group", and is a model for the Lawvere theory of abelian groups. Be warned, however, that this is *not* the same as a grouplike $\mathbb{E}_\infty$ object (hence some people's usage of the word "strict" to distinguish it).

Theorem 2.3. *If A is a complex-periodic $\mathbb{E}_\infty$-ring, the "categorified homology" coalgebra $C_*(K(M^\vee, 2); A)$ is smooth for any $M \in \mathrm{Lat}$. Thus $M \mapsto \mathrm{cSpec}\, C_*(K(M^\vee, 2); A)$ is a product-preserving functor $\mathrm{Lat}^{op} \to \mathrm{Hyp}(A)$, i.e. a formal group.*

Definition 2.4. This is called the *Quillen formal group* of A, $\widehat{\mathbb{G}}_A^Q$.

This name is justified by the following proposition.

Proposition 2.5. *The associated classical formal group is* $\widehat{\mathbb{G}}_A^{Q_0}$.

Proof. Because A is complex-periodic, we have $\widehat{\mathbb{G}}_A^{Q_0} = \operatorname{Spf} A^0(\mathbb{CP}^\infty) = \operatorname{cSpec} A_0(\mathbb{CP}^\infty)$. Since the group structure comes from $\mathbb{CP}^\infty$, its associated abelian functor of points $\mathrm{Lat}^{op} \to \mathrm{Hyp}(\pi_0 A)$ is $\mathbb{Z}^n \mapsto \operatorname{cSpec} A_0((\mathbb{CP}^\infty)^n) \cong \operatorname{cSpec} A_0(K((\mathbb{Z}^n)^\vee, 2)) = \pi_0 C_*(K((\mathbb{Z}^n)^\vee, 2); A)$. The multiplication on $\mathbb{CP}^\infty$ comes from $\mathbb{Z}$, so the abelian structure is compatible with this isomorphism. □

3. Dualizing Lines

Okay, so we've lifted the theory of formal groups to spectral algebraic geometry. But what do we actually get from this generalization?

Answer: A theory of orientations! But to describe it, we'll first need the ability to linearize our formal groups.

Normally, we would do this using the cotangent space. This works fine for discrete formal groups, but the "obvious" spectral analogue, the *cotangent fiber* $\eta^* L_{X/R}$ (constructed from the cotangent complex $L_{X/R}$ by pulling back alone the basepoint η), has some issues:

i) Suppose R is a classical commutative ring. Then $\pi_0(\eta^* L_{X/R}) \cong T^*_{X,\eta}$, but $\eta^* L_{X/R}$ has nonzero π_n for some $n > 0$ unless R is rational.
ii) Unless our $\mathbb{E}_\infty$-ring R is rational, the module $\eta^* L_{X/R}$ need not be projective.

To solve these problems, we choose a different generalization: the *dualizing line*. Let X be a one-dimensional formal hyperplane with basepoint η. The idea is to define an invertible sheaf which is the relative dualizing complex of the canonical projection $X \to \operatorname{Spec} R$ in the sense of Grothendieck-Serre duality, then take the fiber over η. Because this duality theory is hard, however, we (and Lurie) give a simpler definition, reminiscent of the I/I^2 definition in commutative algebra.

Definition 3.1. Let $\varepsilon : \mathcal{O}_X \to R$ classify the basepoint $\eta \in X(\tau_{\geq 0} R)$, and write $\mathcal{O}_X(-\eta) = \operatorname{fib}(\varepsilon)$. The *dualizing line* of (X, η) is the module $\omega_{X,\eta} = R \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\eta)$.

Theorem 3.2 (Properties of ω)**.**

i) $\omega_{X,\eta}$ *is a locally free R-module of rank* 1.
ii) For any $M \in \mathrm{Mod}_R$, *we have a natural equivalence* $\mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, M) \simeq \operatorname{fib}\Big(\mathrm{Map}_{\mathrm{Alg}_R}(\mathcal{O}_X, R \oplus M) \to \mathrm{Map}_{\mathrm{Alg}_R}(\mathcal{O}_X, R)\Big)$, *where the fiber is taken over* ε.
iii) We have a natural fiber sequence of R-modules

$$\Sigma(\omega_{X,\eta}) \to R \otimes_{\mathcal{O}_X} R \xrightarrow{m} R,$$

where m is multiplication on R viewed as an $\mathcal{O}_X$-algebra via ε.

Remark 3.3. The cotangent fiber satisfies a property similar to (ii), where Alg_R is replaced by by CAlg_R. So ω is kind of like a noncommutative version of $\eta^* L$, measuring square-zero extensions of $\mathbb{E}_1$-rings deforming η rather than $\mathbb{E}_\infty$-rings.

Now, let $A \in \mathrm{CAlg}_R$. Then, after replacing A and R with their connective covers, we get a map

$$\begin{aligned}\Omega X(A) &\simeq \mathrm{Map}_{\mathrm{CAlg}_R}(R \otimes_{\mathcal{O}_X} R, A) \\ &\to \mathrm{Map}_R(R \otimes_{\mathcal{O}_X} R, A) \\ &\to \mathrm{Map}_R(\Sigma(\omega_{X,\eta}), A) \simeq \Omega \mathrm{Map}_R(\omega_{X,\eta}, A).\end{aligned}$$

The composite $\mathfrak{L} : \Omega X(\tau_{\geq 0} A) \to \Omega \mathrm{Map}_R(\omega_{X,\eta}, A)$ is called the *linearization map*. Note that it turns maps into our hyperplane into maps out of a version of the cotangent space, as we would expect a linearization procedure to do. (It wouldn't be too far off to call this a contravariant derivative for loops in the space of functions, if you want to sound fancy. In fact, it can completely classify ramification of morphisms; but that's a story for another day.)

Consider now the Quillen formal group of a complex-periodic $\mathbb{E}_\infty$-ring A. We have a fiber sequence

$$\begin{array}{ccccc} \Sigma(\omega_{\widehat{\mathbb{G}}^Q_A}) & \longrightarrow & A \otimes_{C^*(\mathbb{CP}^\infty; A)} A & \xrightarrow{m} & A \\ & & \wr\!\parallel & \nearrow_{ev_1} & \\ & & C^*(S^1, A), & & \end{array}$$

so $\Sigma(\omega_{\widehat{\mathbb{G}}^Q_A}) \simeq C^*_{\mathrm{red}}(S^1, A) \simeq \Sigma^{-1} A$, and finally

$$\boxed{\omega_{\widehat{\mathbb{G}}^Q_A} \simeq \Sigma^{-2} A.}$$

Since ω is the linearization of $\widehat{\mathbb{G}}^Q_A$, this tells us that, intuitively, the Quillen formal group consists of power series in $\pi_2 A$. This result will allow us to relate orientations to the Quillen formal group and view them as a generalization of Bott periodicity.

4. Orientations

Definition 4.1. A *preorientation* of a pointed formal hyperplane X is a pointed map $e : S^2 \to X(\tau_{\geq 0} R)$. We write $\mathrm{Pre}(X) = \Omega^2 X(\tau_{\geq 0} R)$ for the space of preorientations.

Recall that we have $\mathfrak{L} : \Omega X(\tau_{\geq 0} R) \to \Omega \mathrm{Map}_R(\omega_{X,\eta} R)$. Looping this gives a map $\mathrm{Pre}(X) \to \mathrm{Map}_R(\omega_{X,\eta}, \Sigma^{-2} R)$. For $e \in \mathrm{Pre}(X)$, we call its image $\beta_e : \omega_{X,\eta} \to \Sigma^{-2} R$ the *Bott map* associated to e.

Definition 4.2. A preorientation is called an *orientation* if β_e is an equivalence.

The isomorphism described at the end of the last section can be described as the Bott map associated to a certain preorientation. Remember, the Quillen formal group has underlying formal hyperplane $C^*(K(\mathbb{Z}, 2); A)$, so a preorientation of it is just an element of $A^2(\mathbb{CP}^\infty)$. Tracing through the linearization map, we find that the chosen complex orientation c_1 will go to the desired isomorphism. So orientations of the Quillen formal group correspond to complex orientations! In fact, this goes further: it is the universal example of a pre-oriented formal group.

Theorem 4.3. *Let R be a complex-periodic $\mathbb{E}_\infty$-ring, and let $\widehat{\mathbb{G}}$ be a formal group over R. Then we have an equivalence* $\mathrm{Pre}(\widehat{\mathbb{G}}) \simeq \mathrm{Map}_{\mathrm{FGroup}}(\widehat{\mathbb{G}}^Q_R, \widehat{\mathbb{G}})$.

Proof. The computation is similar to the one described above. Note that we have $\mathrm{Pre}(\widehat{\mathbb{G}}) = \mathrm{Map}_*(S^2, \Omega^\infty \widehat{\mathbb{G}}(\tau_{\geq 0} R)) \simeq \mathrm{Map}_{\mathbb{Z}}(\mathbb{CP}^\infty, \widehat{\mathbb{G}}(\tau_{\geq 0} R))$, because $\mathbb{CP}^\infty$ is the free $\mathbb{Z}$-module on S^2. Writing C for the abelian group coalgebra with $\mathrm{cSpec}\,(C) = \widehat{\mathbb{G}}$, we get

$$\begin{aligned}\mathrm{Pre}(\widehat{\mathbb{G}}) &\simeq \mathrm{Map}_{\mathbb{Z}}(\mathbb{CP}^\infty, \mathrm{Map}_{\mathrm{cCAlg}_R}(R, C)) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{cCAlg}_R)}(C_*(\mathbb{CP}^\infty; R), C) \\ &\simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\mathrm{cSpec}\,(C_*(\mathbb{CP}^\infty; R)), \mathrm{cSpec}\,(C)) \\ &\simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbb{G}}^Q_R, \widehat{\mathbb{G}}).\end{aligned}$$

□

We can imitate the linearization computation above to show that not only is $\widehat{\mathbb{G}}^Q_R$ the initial preoriented formal group, it is also the unique oriented formal group.

Theorem 4.4. *A preorientation is an orientation if and only if*

i) R is complex-periodic, and

ii) The induced map $\widehat{\mathbb{G}}^Q_R \to \widehat{\mathbb{G}}$ is an equivalence.

Example. Consider the special case $R = KU$, $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_m$. A computation I'll skip here shows that over $\mathbb{S}$, and thus over any ring R, the dualizing line of $\widehat{\mathbb{G}}_m$ can be canonically identified with R. If we take $e : S^2 \cong \mathbb{CP}^1 \to \Omega^\infty \widehat{\mathbb{G}}_m(KU)$ to be the preorientation corresponding to the tautological bundle $\mathcal{O}(1)$ on $\mathbb{CP}^1$, β_e is the standard Bott isomorphism $KU \xrightarrow{\sim} \Sigma^{-2} KU$. In this sense, the complex orientation of KU is the same information as Bott periodicity. (Likewise, it allows us to interpret complex orientations of other ring spectra as a kind of generalized Bott periodicity.) Combining this with classical Bott periodicity and the theory of oriented deformation rings provides an elegant new proof of Snaith's theorem in the $\mathbb{E}_\infty$ setting.

References

[1] Doron Grossman-Naples. *Complex Orientations Are Partial Strictifications of the Unit*. Nov. 21, 2023. DOI: `10.48550/arXiv.2311.01663`. arXiv: `2311.01663 [math]`. preprint.

[2] Jacob Lurie. *Elliptic Cohomology II: Orientations*. Apr. 2018. URL: `https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf`. preprint.